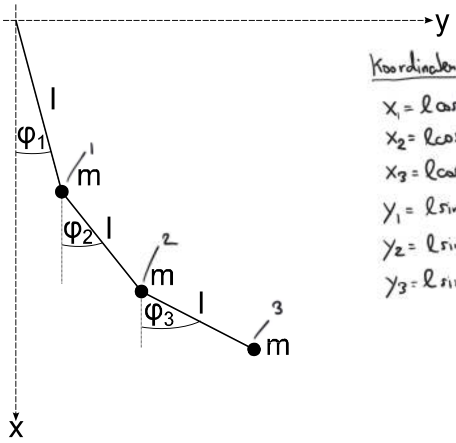
2.22 Beispiel L22

Gegeben sei das ebene Tripelpendel laut Skizze.



Koordinaten:

$x_1 = l\cos\varphi_1$

$x_2 = l\cos\varphi_1 + l\cos\varphi_2$

$x_3 = l\cos\varphi_1 + l\cos\varphi_2 + l\cos\varphi_3$

$y_1 = l\sin\varphi_1$

$y_2 = l\sin\varphi_1 + l\sin\varphi_2$

$y_3 = l\sin\varphi_1 + l\sin\varphi_2 + l\sin\varphi_3$

Bestimmen Sie

(a) die Lagrangefunktion des System.

(b) die Bewegungsgleichungen in allen generalisierten Koordinaten.

Koordinaten:

$x_1 = l\cos\varphi_1$

$x_2 = l\cos\varphi_1 + l\cos\varphi_2$

$x_3 = l\cos\varphi_1 + l\cos\varphi_2 + l\cos\varphi_3$

$y_1 = l\sin\varphi_1$

$y_2 = l\sin\varphi_1 + l\sin\varphi_2$

$y_3 = l\sin\varphi_1 + l\sin\varphi_2 + l\sin\varphi_3$

Geschwindigkeiten:

$\dot{x}_1 = -l\dot{\varphi}_1\sin\varphi_1$

$\dot{x}_2 = -l\dot{\varphi}_1\sin\varphi_1 - l\dot{\varphi}_2\sin\varphi_2$

$\dot{x}_3 = -l\dot{\varphi}_1\sin\varphi_1 - l\dot{\varphi}_2\sin\varphi_2 - l\dot{\varphi}_3\sin\varphi_3$

$\dot{y}_1 = l\dot{\varphi}_1\cos\varphi_1$

$\dot{y}_2 = l\dot{\varphi}_1\cos\varphi_1 + l\dot{\varphi}_2\cos\varphi_2$

$\dot{y}_3 = l\dot{\varphi}_1\cos\varphi_1 + l\dot{\varphi}_2\cos\varphi_2 + l\dot{\varphi}_3\cos\varphi_3$

Quadrate:

$\dot{x}_1^2 = l^2\dot{\varphi}_1^2\sin^2\varphi_1$

$\dot{x}_2^2 = l^2\dot{\varphi}_1^2\sin^2\varphi_1 + 2l^2\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_1\sin\varphi_2 + l^2\dot{\varphi}_2^2\sin^2\varphi_2$

$\dot{x}_3^2 = l^2\dot{\varphi}_1^2\sin^2\varphi_1 + l^2\dot{\varphi}_2^2\sin^2\varphi_2 + l^2\dot{\varphi}_3^2\sin^2\varphi_3 + 2l^2\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_1\sin\varphi_2 + 2l^2\dot{\varphi}_1\dot{\varphi}_3\sin\varphi_1\sin\varphi_3 + 2l^2\dot{\varphi}_2\dot{\varphi}_3\sin\varphi_2\sin\varphi_3$

$\dot{y}_1^2 = l^2\dot{\varphi}_1^2\cos^2\varphi_1$

$\dot{y}_2^2 = l^2\dot{\varphi}_1^2\cos^2\varphi_1 + 2l\dot{\varphi}_1\dot{\varphi}_2\cos\varphi_1\cos\varphi_2 + l^2\dot{\varphi}_2^2\cos^2\varphi_2$

$\dot{y}_3^2 = l^2\dot{\varphi}_1^2\cos^2\varphi_1 + l^2\dot{\varphi}_2^2\cos^2\varphi_2 + l^2\dot{\varphi}_3^2\cos^2\varphi_3 + 2l^2\dot{\varphi}_1\dot{\varphi}_2\cos\varphi_1\cos\varphi_2 + 2l^2\dot{\varphi}_1\dot{\varphi}_3\cos\varphi_1\cos\varphi_3 + 2l^2\dot{\varphi}_2\dot{\varphi}_3\cos\varphi_2\cos\varphi_3$

Energien:

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\right)$$

$$= \frac{ml^2}{2}\Big[\dot{\varphi}_1^2\left(3\sin^2\varphi_1 + 3\cos^2\varphi_1\right) + \dot{\varphi}_2^2\left(2\sin^2\varphi_2 + 2\cos^2\varphi_2\right) + \dot{\varphi}_3^2\left(\sin^2\varphi_3 + \cos^2\varphi_3\right)$$
$$+ 2\dot{\varphi}_1\dot{\varphi}_2\left(2\sin\varphi_1\sin\varphi_2 + 2\cos\varphi_1\cos\varphi_2\right) + 2\dot{\varphi}_1\dot{\varphi}_3\left(\sin\varphi_1\sin\varphi_3 + \cos\varphi_1\cos\varphi_3\right) + 2\dot{\varphi}_2\dot{\varphi}_3\left(\sin\varphi_2\sin\varphi_3 + \cos\varphi_2\cos\varphi_3\right)$$

$$T = \frac{ml^2}{2}\left[3\dot{\varphi}_1^2 + 2\dot{\varphi}_2^2 + \dot{\varphi}_3^2 + 4\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1-\varphi_2) + 2\dot{\varphi}_1\dot{\varphi}_3\cos(\varphi_1-\varphi_3) + 2\dot{\varphi}_2\dot{\varphi}_3\cos(\varphi_2-\varphi_3)\right]$$

$$V = -mg(x_1 + x_2 + x_3) = -mgl\left(3\cos\varphi_1 + 2\cos\varphi_2 + \cos\varphi_3\right)$$

$$\mathcal{L} = \frac{ml^2}{2}\left[3\dot{\varphi}_1^2 + 2\dot{\varphi}_2^2 + \dot{\varphi}_3^2 + 4\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1-\varphi_2) + 2\dot{\varphi}_1\dot{\varphi}_3\cos(\varphi_1-\varphi_3) + 2\dot{\varphi}_2\dot{\varphi}_3\cos(\varphi_2-\varphi_3)\right]$$
$$+ mgl\left(3\cos\varphi_1 + 2\cos\varphi_2 + \cos\varphi_3\right)$$

Bewegungsgleichungen:

φ_1: $d_t(\partial_{\dot{\varphi}_1}\mathcal{L}) - \partial_{\varphi_1}\mathcal{L} = 0$

$$d_t\left[\frac{ml^2}{2}\left(6\dot{\varphi}_1 + 4\dot{\varphi}_2\cos(\varphi_1-\varphi_2) + 2\dot{\varphi}_3\cos(\varphi_1-\varphi_3)\right)\right] - \left[\frac{ml^2}{2}\left(-4\dot{\varphi}_1\dot{\varphi}_2\sin(\varphi_1-\varphi_2) - 2\dot{\varphi}_1\dot{\varphi}_3\sin(\varphi_1-\varphi_3)\right)\right.$$
$$\left. + mgl(-3\sin\varphi_1)\right] = 0$$

$$\frac{ml^2}{2}\Big(6\ddot{\varphi}_1 + 4\ddot{\varphi}_2\cos(\varphi_1-\varphi_2) - 4\dot{\varphi}_2\sin(\varphi_1-\varphi_2)\dot{\varphi}_1 - 4\dot{\varphi}_2\sin(\varphi_1-\varphi_2)(-\dot{\varphi}_2) + 2\ddot{\varphi}_3\cos(\varphi_1-\varphi_3) - 2\dot{\varphi}_3\sin(\varphi_1-\varphi_3)\dot{\varphi}_1$$
$$- 2\dot{\varphi}_3\sin(\varphi_1-\varphi_3)(-\dot{\varphi}_3)\Big) + 2ml^2\dot{\varphi}_1\dot{\varphi}_2\sin(\varphi_1-\varphi_2) + ml^2\dot{\varphi}_1\dot{\varphi}_3\sin(\varphi_1-\varphi_3) + 3mgl\sin\varphi_1 = 0$$

$$3\ddot{\varphi}_1 + 2\ddot{\varphi}_2\cos(\varphi_1-\varphi_2) + \ddot{\varphi}_3\cos(\varphi_1-\varphi_3) + 2\dot{\varphi}_2^2\sin(\varphi_1-\varphi_2) + \dot{\varphi}_3^2\sin(\varphi_1-\varphi_3) + \frac{3g}{l}\sin\varphi_1 = 0$$

φ_2: $d_t(\partial_{\dot{\varphi}_2}\mathcal{L}) - \partial_{\varphi_2}\mathcal{L} = 0$

$$2\ddot{\varphi}_2 + 2\ddot{\varphi}_1\cos(\varphi_1-\varphi_2) + \ddot{\varphi}_3\cos(\varphi_2-\varphi_3) - 2\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2) + \dot{\varphi}_3^2\sin(\varphi_2-\varphi_3) + \frac{2g}{l}\sin\varphi_2 = 0$$

φ_3: $d_t(\partial_{\dot{\varphi}_3}\mathcal{L}) - \partial_{\varphi_3}\mathcal{L} = 0$

$$\ddot{\varphi}_3 + \ddot{\varphi}_1\cos(\varphi_1-\varphi_3) + \ddot{\varphi}_2\cos(\varphi_2-\varphi_3) - \dot{\varphi}_1^2\sin(\varphi_1-\varphi_3) - \dot{\varphi}_2^2\sin(\varphi_2-\varphi_3) + \frac{g}{l}\sin\varphi_3 = 0$$